Downward Closed Topology On ℕ With Its Effect On The Levels Of ℕ

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Submission date : 22-12-2022 Acceptance date : 27-2-2023 Electronic publishing date: 28-02-2023

Abstract: Since many facts about the natural numbers ℕ in the number theory had been established with the usual divisibility that is defined on ℕ. Then to connect these facts with some topological properties on ℕ for some topologies that are defined on ℕ, some topologies, that depend on the divisibility, have been defined. In this paper, we study a topology that is defined on ℕ and also depending on the usual divisibility, this topology is the topology that contains the set of all downward closed subsets of ℕ and it is called downward closed topology on ℕ. Furthermore, The relation between the divisibility and the limit points, and the topological relation between the levels of ℕ have discussed.

Keywords: Downward closed topology, Natural numbers set, divisibility.

1- Introduction

In [5], we have used the upward closed topology that is defined on ℕ to connect the usual divisibility that is defined on ℕ with the limit points. Furthermore, the relation between the divisibility and the limit points, and the topological relation between the levels of ℕ, which is the numbers that are in the up levels are limit points to the lower levels, have studied.

In this work, we introduce a topology and study The relation between the divisibility and the limit points, and the topological relation between the levels of ℕ.

(a) Numbers concepts

If the set of natural numbers ℕ has been given. We say that $a \in \mathbb{N}$ divide $b \in \mathbb{N}$ if there is $c \in \mathbb{N}$ such that $b = a.c$ and we denote that by $a|b$. For all $a, b \in \mathbb{N}$ we say that $c \in \mathbb{N}$ is the greatest divisor of a and b if $c|a$, $c|b$ and if $d|a$, $d|b$ then $d \leq c$, and we denote that by $(a, b) = c$. If $a \in \mathbb{N}$ and a is divided by 1 and itself, then we say that a is prime number and we denote the set of prime numbers by P. If $(a, b) = 1$, $a, b \in \mathbb{N}$ we say that a and b are relative prime.

Lemma 1.1 [2] Every $n \in \mathbb{N}, n > 1$, is divisible by a prime number.

Theorem 1.1 The Unique Factorization Theorem [2]
Any natural number greater than one can be written as a product of primes in one and only one way. i.e. for any $n \in \mathbb{N}, n > 1$ can be written in exactly one way in the form
$n = {p_1}^{e_1}{p_2}^{e_2} \dots \dots . {p_k}^{e_k}$ where, $e_i \geq 0$, $p_i \in P, i = 1, 2, \dots \dots, k$, and $p_i \neq p_j$.

We call this representation by the Prime-Power Decomposition of n.

One of the results to the Unique Factorization Theorem, we can divide the set of natural numbers into infinitely many levels L_i, where $i \geq 0$ such that $L_0 = \{1\}$,
$L_i = \{a_1.a_2 \dots\dots a_i : a_1, a_2, \dots \dots, a_i \in P\}$ where $i \geq 1$.
$\mathbb{N} = \bigcup_{i=0}^{\infty} L_i$ and $L_i \cap L_j = \emptyset$, where $i \neq j$ ($\bigcap_{i=0}^{\infty} L_i = \emptyset$).

For any $a \in \mathbb{N}$,
$a \uparrow = \{ n \in \mathbb{N} : a|n \} = \{ma : m \in \mathbb{N}\}$

The collection of upward closed subset of $\mathbb{N}$ is $\mu = \{A \subseteq \mathbb{N} : A = A\uparrow\}$ where $A\uparrow = \{n \in \mathbb{N} : \exists\, a \in A, a|n\}$ For more details, see [4].

(b) Topological Concepts[3,6]

If X is any set, the collection τ of subsets of X is called topology on X if satisfying the following conditions: the empty set $\emptyset$ and X belong to τ, arbitrary union of elements of τ belongs to τ, and any finite intersection of elements of τ belongs to τ. We say that (X, τ) is a topological space. A subset U of X is called an open set if $U \in \tau$, and a subset F of X is called closed if $X - F \in \tau$.

A collection ß $\subset \tau$ is called a base for a topological space (X, τ) if $\tau = \{\bigcup_{B\in\eta} B : \eta \subset ß\}$.

A limit point of a set A in a topological space X is a point $x \in X$ such that each open set contained x also contains some points of A other than x. i.e. $A \cap (U - \{x\}) = \emptyset$, for any open set U, $x \in$ U, the limit points of a set A denote by $\acute{A}$. A set A is closed if and only if contains all of its limit points. Closure of a set A is denoted by $\bar{A}$, and defined by $\bar{A} = A \cup \acute{A}$. The set A in a topological space X is dense if $\bar{A} = $ X.

Lemma1.2 [5] The collection of upward closed subsets of $\mathbb{N}$ defined topology on $\mathbb{N}$ i with the base ß = $\{n\uparrow : n \in \mathbb{N}\}$.
We denote to a topological space $\mathbb{N}$ with μ by $(\mathbb{N}, \mu)$.

Lemma 1.3 [5] In the space $(\mathbb{N}, \mu)$ of upward closed subset of $\mathbb{N}$:

(a) a is a limit point for {b} if and only if $a|b, a \neq b$.

(b) $\{a\}' = \{n \in \mathbb{N}: n|a, n \neq a\}$, for any $a \in \mathbb{N}$.

(c) n is a limit point for A if and only if there exists $a \in A$, $n|a$, and $n \neq a$.

(d) $A' = \{n \in \mathbb{N}: \exists a \in A, n|a, n \neq a\}$ for any $A \subseteq \mathbb{N}$.

2- Downward closed subsets of ℕ

Some of the subsets of $\mathbb{N}$ contain all the numbers that are divide their numbers, these sets are called downward closed subsets of $\mathbb{N}$ and the set that contains all of the downward closed subsets of $\mathbb{N}$ is called the collection of downward closed subsets of $\mathbb{N}$ as it will be defined in the following definition.

Definition 2.1 [1] **(a)** For any $a \in \mathbb{N}$,

$$a\downarrow = \{n \in \mathbb{N}: n|a\}.$$

(b) The collection of downward closed subset of $\mathbb{N}$ is $\nu = \{A \subseteq \mathbb{N}: A = A\downarrow\}$ where $A\downarrow = \{n \in \mathbb{N}: \exists\, a \in A, n|a\}$.

Example 2.1 $1\downarrow = \{1\}, 3\downarrow = \{1,3\}$,
$P\downarrow = P \cup \{1\}, \{1,2\}\downarrow = \{1,2\}$,
$(P \cup \{1\})\downarrow = (P \cup \{1\})$

Lemma 2.1 (a) $A \subseteq A\downarrow$, for any $A \subseteq \mathbb{N}$.
(b) $1 \in A$ for any $A \in \nu, A \neq \emptyset$.
(c) $\emptyset, \mathbb{N} \in \nu$
(d) $L_i \notin \nu$ for all $i \geq 1$.
Proof: **(a)** Obvious.
(b) Obvious.
(c) It's clear that is $\mathbb{N} \in \nu$.
If there exists $n \in \emptyset\downarrow$, then there exists $a \in \emptyset$ such that $n|a$, but this is impossible, so $n \notin \emptyset\downarrow$ and $\emptyset = \emptyset\downarrow$.
(d) $L_i\downarrow = \bigcup_{j=0}^{i} L_j$ for all $i \geq 1$. ■

Lemma 2.2 **(a)** $A^c \in \mu$ for any $A \in \nu$.
(b) $A^c \in \nu$ for any $A \in \mu$.
Proof: **(a)** Let $A \subseteq \mathbb{N}$, $A \in \nu$. If $A = \emptyset$ or $\mathbb{N}$ then $A^c = \mathbb{N}$ or $\emptyset$, so $A^c \in \mu$. If $A \neq \emptyset$, $\mathbb{N}$, let $n \in \mathbb{N}$, $a|n$ for some $a \in A^c$, and if $n \in A = A\downarrow$, then there exists $b \in A$, $n|b$, so $a|b$, $a \in A$, so we have a contradiction. Thus $n \in A^c$, $A^c = A^c\uparrow$ and $A^c \in \mu$
(b) Similar to (a) ■

Lemma 2.3 **(a)** $n\downarrow, A\downarrow$ are downward closed sets for any $n \in \mathbb{N}, A \subseteq \mathbb{N}$
(b) If A is downward closed then $A\downarrow = (A\downarrow)\downarrow$.
Proof: **(a)** Obvious.
(b) Let $A \in \nu$,
$(A\downarrow)\downarrow = \{n \in \mathbb{N}: \exists\, a \in A\downarrow, n|a\}$
$= \{n \in \mathbb{N}: \exists\, a \in A, n|a\} = A\downarrow$.■

Lemma 2.4 (a) If n_1, $n_2 \in$ ℕ, $n_1|n_2$, then for any $A\in v$ contains n_2 contains also n_1
(b) If n_1, $n_2 \in$ ℕ, $n_1|n_2$ then $n_1 \downarrow\subseteq n_2\downarrow$
Proof: (a) Let n_1, $n_2 \in$ ℕ, $n_1|n_2$, and let $A \in v$, $n_2 \in A$, so there exists $a \in A$, $n_2|a$, and $n_1|a$. Thus $n_1 \in A$.
(b) Let n_1, $n_2 \in$ ℕ, $n_1|n_2$, and let $n \in n_1\downarrow$, so $n|n_1$, and $n|n_2$. Thus $n \in n_2\downarrow$, and $n_1\downarrow \subseteq n_2\downarrow$. ■

Lemma 2.5 (a) If $A \subseteq B$, then $A \downarrow\subseteq B\downarrow$ for any A, $B \subseteq$ ℕ.
(b) If A, $B \in v$, then $A \cap B \in v$, ($\bigcap_{i=1}^{n} A_i \in v$, where $A_i \in v$, i = 1, 2,.., n).
(c) If A, $B \in v$, then $A \cup B \in v$, ($\bigcup_{i=1}^{n} A_i \in v$, where $A_i \in v$, i = 1, 2,.., n).
(d) $\bigcup_{i=1}^{\infty} A_i \in v$, where $A_i \in v$, i = 1,2,…
Proof: (a) Let A, $B \subseteq$ ℕ, $A \subseteq B$, $n \in$ ℕ, $n \in A\downarrow$, so there exists $a \in A$, $n|a$, so $a \in B$, $n \in B\downarrow$. Thus $A \downarrow\subseteq B\downarrow$.
(b) Let A, $B \in v$, then

$$(A \cap B)\downarrow=\{n \in ℕ: \exists\, a \in A \cap B, n|a\}$$
$$=\{n \in ℕ:\exists\, a \in A, n|a\}\cap\{n\in ℕ\ \exists\, a \in B, n|a\}$$
$$= A\downarrow \cap B\downarrow= A \cap B.$$

(c) Let A, $B \in v$, then

$$(A \cup B)\downarrow=\{n\in ℕ: \exists\, a \in A \cup B, n|a\}$$
$$= \{n\in ℕ: \exists\, a \in A, n|a\} \cup \{ n\in ℕ: \exists\, a \in B, n|a\}$$
$$= A\downarrow \cup B\downarrow= A \cup B$$

(d) Similar to (c) ■

Theorem 2.1 (a) If $A \subseteq$ ℕ is downward closed set, then $A=\bigcup_{a\in A} a\downarrow$.
(b) $v = \{ \bigcup_{a\in A} a\downarrow: A \in v\}$.
Proof: (a) Let $A \subseteq$ ℕ is downward closed set, and $n \in A$, then there exists $a \in A$ such that $n|a$, so $n \in a\downarrow$, and $n \in \bigcup_{a\in A} a\downarrow$. Thus $A \subseteq \bigcup_{a\in A} a\downarrow$.
On the other hand, if n $\in \bigcup_{a\in A} a\downarrow$, then there exists $a \in A$, $n \in a\downarrow$, so $n|a$, and $n\in A\downarrow$. Thus $\bigcup_{a\in A} a \downarrow\subseteq A\downarrow=A$. Therefore $A = \bigcup_{a\in A} a\downarrow$
(b) By (a) and definition of downward closed ■

Theorem 2.2 $\bigcup_{i=0}^{k} \mathrm{L}_i$, where $k = 0,1,2,\ldots$ are downward closed subsets of ℕ.

Proof : Let L_j be any level of ℕ, where $j > k$, and let $n \in \mathrm{L}_j$ then, $n = {a_1}^{n_1} {a_2}^{n_2} \ldots. {a_j}^{n_j}$ where, $n_1 + n_2 + \cdots + n_j = j$, and $a_1, a_2, \ldots\ldots, a_j \in$ P, so n can't divide any number in L_i , where $i \leq k$. Thus $(\bigcup_{i=0}^{k} \mathrm{L}_i)\downarrow=\bigcup_{i=0}^{k} \mathrm{L}_i$ for all k = 0,1,2,.....
■

3- Downward Closed Topology On ℕ

The collection of downward closed subsets of ℕ define topology on ℕ. This topology gives a connection between the usual divisibility that is defined on ℕ and the limit points in way similar to the way that gave in the upward closed topology that was defined on ℕ. Moreover, in ℕ with this topology the numbers that are in the up levels are limit points to the lower levels.

Lemma 3.1 (a) The collection of downward closed subsets of ℕ define topology on ℕ.
(b) The collection ß = $\{n\downarrow: n \in ℕ\}$ is a basis for v.
Proof: (a) Let $v = \{A \subseteq ℕ: A = A\downarrow\}$ where $A\downarrow= \{ n \in ℕ : \exists\, a \in A, n|a \}$.
(1) Since ℕ↓= ℕ, $\emptyset=\emptyset\downarrow$, then ℕ, $\emptyset \in v$
(2) If $A_1, A_2, \ldots, A_n \in v$,then by (Lemma (2.5)(b)) $\bigcap_{i=1}^{n} A_i \in v$
(3) Let $A_\gamma \in v, \gamma \in \Gamma$, then $A_\gamma\downarrow= A_\gamma$ $\forall\, \gamma \in \Gamma$ and

$$(\bigcup_{\gamma\in\Gamma} A_\gamma)\downarrow= \{n \in ℕ: \exists\, a\ \bigcup_{\gamma\in\Gamma} A_\gamma , n|a\}$$
$$=\bigcup_{\gamma\in\Gamma}\{n \in ℕ: \exists\, a \in A_\gamma , n|a\}$$
$$=\bigcup_{\gamma\in\Gamma}(A_\gamma\downarrow) = \bigcup_{\gamma\in\Gamma} A_\gamma$$

So$\bigcup_{\gamma\in\Gamma} A_\gamma\in v$. Thus, v defined topology on ℕ
(b) By (Theorem (2.1) (a)) for any $A \in v$, $A = \bigcup_{n\in A} n\downarrow$.Thus ß is a basis for v. ■

We denote to a topological space ℕ with v by (ℕ , v).

Lemma 3.2 In the space ℕ with downward closed topology.
(a) a is a limit point for $\{b\}$ if and only if $b|a$, $a \neq b$.

(b) $\{a\}' = \{n \in \mathbb{N} : a|n,\ n \neq a\}$, for any $a \in \mathbb{N}$.
(c) n is a limit point for A if and only if there exists $a \in A$, $a|n$, $n \neq a$.
(d) $A' = \{n \in \mathbb{N} : \exists\, a \in A,\ a|n,\ n \neq a\}$ for any $A \subseteq \mathbb{N}$.

Proof: (a) (⇒) Let $a, b \in \mathbb{N}$, a is a limit point to $\{b\}$, since $a\downarrow$ is an open set, $a \in a\downarrow$. So $\{b\} \cap (a\downarrow - \{a\}) \neq \emptyset$, and $b \in a\downarrow$. Thus $b|a$, $a \neq b$.
(⇐) Let $a, b \in \mathbb{N}$, $b|a$, $a \neq b$, and let U is an open set, $a \in U$, by (Lemma (2. 4)(a)) we have $b \in U$, so $\{b\} \cap (U - \{a\}) \neq \emptyset$. Thus, a is a limit point to $\{b\}$.
(b) By (a) If $n \in \mathbb{N}$, $a|n$, $n \neq a$, then n is a limit point to $\{a\}$. Thus $\{a\}' = \{n \in \mathbb{N} : a|n,\ n \neq a\}$.
(c) (⇒) Let $n \in \mathbb{N}$, $A \subseteq \mathbb{N}$, n is a limit point to A, since $n\downarrow$ is an open set, $n \in n\downarrow$, so $A \cap (n\downarrow - \{n\}) \neq \emptyset$, and there exists $a \in A$, $n\downarrow$, $a|n$, $n \neq a$.
(⇐) Let $A \subseteq \mathbb{N}$, $a \in A$, $n \in \mathbb{N}$, $a|n$, $n \neq a$, and let U is an open set, $n \in U$. By (Lemma (2.4) (a)) we have $a \in U$, so $A \cap (U - \{n\}) \neq \emptyset$. Thus, n is a limit point for A.
(d) By (c) if $a \in A$, $a|n$, $n \neq a$, then n is a limit point for A. Thus $A' = \{n \in \mathbb{N} : \exists\, a \in A,\ a|n,\ n \neq a\}$ ■

Corollary 3.1 In the space $(\mathbb{N}, \nu)$, for any $n \in \mathbb{N}$, $n > 1$. n is prime number or a limit point to $\{n_i\}$ for some i where n_i is prime number.
Proof: By (Lemma 1-1) and (Lemma 3.2 (a)) ■

Corollary 3.2 In the space $(\mathbb{N}, \nu)$.
(a) If $a|b$, then $\{b\}' \subset \{a\}'$
(b) For any $A \subseteq \mathbb{N}$, A' is upward closed subset of $\mathbb{N}$.
Proof: (a) Let $a|b$, and $n \in \mathbb{N}$, $n \in \{b\}'$, then by (Lemma 3.2 (a)) $b|n$, so $a|n$, and by (Lemma 3.2 (a)) $n \in \{a\}'$. Thus $\{b\}' \subset \{a\}'$.
(b) Let $n \in \mathbb{N}$, $a|n$ for some $a \in A'$, so by (Lemma 3.2 (c)) $n \in A''$, and since $A'' \subseteq A'$, so $n \in A'$. Thus $A' = A'\uparrow$, $A' \in \mu$. ■

Corollary 3.3 In the space $(\mathbb{N}, \mu)$. A' is downward closed subset of $\mathbb{N}$, for any $A \subseteq \mathbb{N}$.
Proof: By (Lemma 1.1 (a)) and similar to (Corollary 3.2 (b)) ■

Corollary 3.4 In the space $(\mathbb{N}, \nu)$. $A \subseteq \mathbb{N}$ is closed if and only if for any $a \in A$, $a|b$. $b \in A$.
Proof:(⇒) Let $A \subseteq \mathbb{N}$ is closed, and $a \in A$, $a|b$. So, $b \in A'$, and since the closed set contains all its limit points, so $b \in A$
(⇐) Let $A \subseteq \mathbb{N}$, $a \in A$, $a|b$, $b \in A$. So $b \in A'$ and since b is an arbitrary number in $\mathbb{N}$, so A contains all its limit points and is closed. ■

Theorem 3.1 In the space $(\mathbb{N}, \nu)$, the numbers that are in the up levels are limit points to the lower levels.
i.e. $L'_i = \bigcup_{j=i+1}^{\infty} L_j$, where $i = 0,1,2,\ldots$
Proof: Let L_i, L_j are two levels of $\mathbb{N}$, where $j > i$ and let $a \in L_j$, so $a = n_1^{e_1} n_2^{e_2} \ldots n_j^{e_j}$ where $n_1, n_2, \ldots, n_j \in P$, $e_1 + e_2 + \ldots + e_j = j$. And, let $n = \acute{n}_1^{\acute{e}_1} \acute{n}_2^{\acute{e}_2} \ldots \acute{n}_i^{\acute{e}_i} \in L_i$, where $\{\acute{n}_1, \acute{n}_2, \ldots, \acute{n}_i\} \subset \{n_1, n_2, \ldots, n_j\}$, $\acute{e}_1 + \acute{e}_2 + \cdots + \acute{e}_i = i$, so $n|a$, and by (Lemma 3.2 (c)) a is a limit point to L_i. If $a \in L_k$, where $k \leq i$, then $a\downarrow$ is an open set and $L_k \cap (a\downarrow - \{a\}) = \emptyset$. Thus a is not a limit point to L_k. So $\acute{L}_i = \bigcup_{j=i+1}^{\infty} L_j$, where $i = 0,1,2,\ldots$ ■

Corollary 3.5 In the space $(\mathbb{N}, \nu)$
(a) $(\bigcup_{i=0}^{k} L_i)' = \mathbb{N} - \{1\}$ for all $k = 0,1,2,\ldots$
(b) $\overline{\bigcup_{i=0}^{k} L_i} = \mathbb{N}$ ($\bigcup_{i=0}^{k} L_i$ are dense for all $k = 0,1,2,\ldots$).

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